M251 Practice Exam 6 9.7-9.10
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1. Find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of f at $x = c$. What is P_1 called? The tangent line.

$f(x) = \frac{2}{\sqrt{x}}, c = 9$ $\quad P_1(x) = f(9) + f'(9)(x-9)$

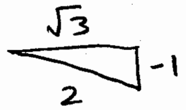
$f(9) = \frac{2}{3}$ $\quad f'(x) = -x^{-\frac{3}{2}}$

$f'(9) = -\frac{1}{27}$ $\quad P_1(x) = \frac{2}{3} - \frac{(x-9)}{27}$

2. Find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of f at $x = c$. What is P_1 called? The tangent line.

$f(x) = \tan x, c = -\frac{\pi}{6}$ $\quad P_1(x) = f\left(-\frac{\pi}{6}\right) + f'\left(-\frac{\pi}{6}\right)\left(x + \frac{\pi}{6}\right)$

$\tan\left(-\frac{\pi}{6}\right) = \frac{-1}{\sqrt{3}}$



$P_1(x) = -\frac{1}{\sqrt{3}} + \frac{4}{3}\left(x + \frac{\pi}{6}\right)$

$f'(x) = \sec^2(x)$ $\quad f'\left(-\frac{\pi}{6}\right) = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$

3. Find the Maclaurin polynomial of degree 3 for the function.

$f(x) = e^{-3x}$ $\quad f(0) = 1$ $\quad 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3$

$f'(x) = -3e^{-3x}$ $\quad f'(0) = -3$

$f''(x) = 9e^{-3x}$ $\quad f''(0) = 9$

$f'''(x) = -27e^{-3x}$ $\quad f'''(0) = -27$

4. Find the Maclaurin polynomial of degree 4 for the function.

$f(x) = e^{11x}$ $\quad f(0) = 1$ $\quad 1 + 11x + \frac{121}{2}x^2 + \frac{1331}{6}x^3 + \frac{14641}{24}x^4$

$f'(x) = 11e^{11x}$ $\quad f'(0) = 11$

$f''(x) = 121e^{11x}$ $\quad f''(0) = 121$

$f'''(x) = 1331e^{11x}$ $\quad f'''(0) = 1331$

$f^{(4)}(x) = 14641e^{11x}$ $\quad f^{(4)}(0) = 14641$

5. Find the Maclaurin polynomial of degree 5 for the function.

$f(x)=\sin(3x)$

$f'(x) = 3\cos(3x)$
$f''(x) = -9\sin(3x)$
$f'''(x) = -27\cos(3x)$
$f^{(4)}(x) = 81\sin(3x)$
$f^{(5)}(x) = 243\cos(3x)$

$f(0)=0$
$f'(0)=3$
$f''(0)=0$
$f'''(0)=-27$
$f^{(4)}(0)=0$
$f^{(5)}(0)=243$

$3x - \frac{27}{6}x^3 + \frac{243}{120}x^5$

6. Find the Maclaurin polynomial of degree 4 for the function.

$f(x)=\cos(x)$
$f'(x) = -\sin(x)$
$f''(x) = -\cos(x)$
$f'''(x) = \sin(x)$
$f^{(4)}(x) = \cos(x)$

$f(0)=1$
$f'(0)=0$
$f''(0)=-1$
$f'''(0)=0$
$f^{(4)}(0)=1$

$1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

7. Find the fourth degree Maclaurin polynomial for the function.

$f(x)=\frac{1}{x+6}$

$f'(x) = \frac{-1}{(x+6)^2}$

$f''(x) = \frac{2}{(x+6)^3}$

$f'''(x) = \frac{-6}{(x+6)^4}$

$f^{(4)}(x) = \frac{24}{(x+6)^5}$

$f(0) = \frac{1}{6}$

$f'(0) = \frac{-1}{36}$

$f''(0) = \frac{2}{216} = \frac{1}{108}$

$f'''(0) = \frac{-1}{216}$

$f^{(4)}(0) = \frac{1}{324}$

$\frac{1}{6} - \frac{x}{36} + \frac{x^2}{216} - \frac{x^3}{1296} + \frac{x^4}{7776}$

8. Find the third degree Taylor polynomial centered at $c = 1$ for the function.

$f(x)=\sqrt[4]{x}$

$f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$

$f''(x) = -\frac{3}{16}x^{-7/4}$

$f'''(x) = \frac{21}{64}x^{-11/4}$

$f(1)=1$

$f'(1)=\frac{1}{4}$

$f''(1)=-\frac{3}{16}$

$f'''(1)=\frac{21}{64}$

$1 + \frac{x-1}{4} - \frac{3(x-1)^2}{32} + \frac{7(x-1)^3}{128}$

9. Find the fourth degree Taylor polynomial centered at $c = 2$ for the function.

$f(x)=\ln x$
$f'(x) = x^{-1}$
$f''(x) = -x^{-2}$
$f'''(x) = 2x^{-3}$
$f^{(4)}(x) = -6x^{-4}$

$f(2)=\ln 2$
$f'(2)=\frac{1}{2}$
$f''(2)=-\frac{1}{4}$
$f'''(2)=\frac{1}{4}$
$f^{(4)}(2)=-\frac{3}{8}$

$\ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64}$

10. Find the radius of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{10^n}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+1}}{10^{n+1}}}{\frac{(-1)^n x^n}{10^n}} \right| < 1$$

$$\lim_{n\to\infty} \left|\frac{x}{10}\right| = \left|\frac{x}{10}\right| < 1 \implies |x| < 10 \qquad \text{radius of conv} = 10$$

11. Find the radius of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{(4x)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(4x)^{2(n+1)}}{(2(n+1))!}}{\frac{(4x)^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \frac{(4x)^2}{(2n+1)(2n+2)} = 0$$

$\implies$ converges for all x $\implies$ radius of conv $= \infty$

12. Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} \left(\frac{x}{7}\right)^n$$

common ratio of geometric series $= \frac{x}{7} \implies \left|\frac{x}{7}\right| < 1 \Rightarrow |x| < 7$

at $x=7$ $\sum \left(\frac{7}{7}\right)^n$ diverges (nth term test) $\implies$ interval of convergence is $(-7,7)$

at $x=-7$ $\sum \left(\frac{-7}{7}\right)^n$ diverges (n^{th} term test)

13. Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} \frac{(6x)^n}{(6n)!}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(6x)^{n+1}}{(6(n+1))!}}{\frac{(6x)^n}{(6n)!}} \right| = \lim_{n\to\infty} \frac{6x}{(6n+1)(6n+2)(6n+3)(6n+4)(6n+5)(6n+6)} = 0$$

< 1

for all x

converges for all x

14. Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-7)^n}{5^n}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)!\,(x-7)^{n+1}}{5^{n+1}}}{\frac{(-1)^n\, n!\,(x-7)^n}{5^n}} \right| = \lim_{n\to\infty} \frac{(n+1)(x-7)}{5} = \infty \text{ for all } x$$

converges only for $x=7$ $\quad [7,7]$

15. Find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\sum_{n=1}^{\infty} \frac{(x-10)^{n-1}}{10^{n-1}}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(x-10)^n}{10^n}}{\frac{(x-10)^{n-1}}{10^{n-1}}} \right| = \lim_{n\to\infty} \left|\frac{x-10}{10}\right| = \left|\frac{x-10}{10}\right| < 1$$

$\Rightarrow |x-10| < 10$ at $x=20$ and $x=0$ the series does not converge (nth term test)

$\Rightarrow$ interval of convergence is $(0,20)$

16. Find the interval of convergence of (i) $f(x)$, (ii) $f'(x)$, (iii) $f''(x)$, and (iv) $\int f(x)dx$ of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{3}{n+1}\left(\frac{x}{3}\right)^{n+1} + C$$

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n}{3}\left(\frac{x}{3}\right)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} \frac{n(n-1)}{9}\left(\frac{x}{3}\right)^{n-2}$$

$$\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$$

	$x=3$	$x=-3$	interval
$f(x)$	$\sum 1$ div	$\sum(-1)^n$ div	$(-3,3)$
$f'(x)$	$\sum \frac{n}{3}$ div	$\sum \frac{n}{3}(-1)^n$ div	$(-3,3)$
$f''(x)$	$\sum \frac{n(n-1)}{9}$ div	$\sum \frac{n(n-1)(-1)^n}{9}$ div	$(-3,3)$
$\int f(x)$	$\sum \frac{3}{n+1}$ div	$\sum \frac{3}{n+1}(-1)^n$ conv	$[-3,3)$

17. Find the interval of convergence of (i) $f(x)$, (ii) $f'(x)$, (iii) $f''(x)$, and (iv) $\int f(x)dx$ of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

$$\lim_{n\to\infty} \left| \frac{\frac{(x-7)^{n+1}}{n+1}}{\frac{(x-7)^n}{n}} \right| = |x-7| < 1$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-7)^n}{n}$$

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-7)^{n-1}$$

$$f''(x) = \sum_{n=2} (-1)^{n+1}(n-1)(x-7)^{n-2}$$

$$\int f(x)dx = \sum_{n=1} \frac{(-1)^{n+1}(x-7)^{n+1}}{n(n+1)} + C$$

	$x=6$	$x=8$	interval
$f(x)$	$\sum \frac{-1}{n}$ div	$\sum \frac{(-1)^{n+1}}{n}$ conv	$(6,8]$
$f'(x)$	$\sum -1$ div	$\sum (-1)^n$ div	$(6,8)$
$f''(x)$	$\sum (-1)^n(n-1)$ div.	$\sum (1-n)$ div.	$(6,8)$
$\int f(x)$	$\sum \frac{1}{n(n+1)}$ conv	$\sum \frac{(-1)^{n+1}}{n(n+1)}$ conv	$[6,8]$

18. Find a geometric power series for the function centered at 0, ~~(i) by the technique shown in Examples 1 and 2 and (ii) by long division.~~

$$f(x)=\frac{6}{8-x} = \frac{3/4}{1-\frac{x}{8}} = \sum_{n=0}^{\infty} \frac{3}{4}\left(\frac{x}{8}\right)^n$$

19. Find a power series for the function, centered at c, and determine the interval of convergence.

$$f(x)=\frac{2}{4+x},\ c=10$$

$$= \frac{2}{4+(x-10)+10}$$

$$\frac{\frac{1}{7}}{1+\frac{x-10}{14}} = \sum_{n=0}^{\infty} \frac{1}{7}\left(\frac{10-x}{14}\right)^n$$

20. Use the power series

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$h(x)=\frac{-12}{x^2-1} = \frac{12}{1-x^2} = 12\left(\frac{1}{1+(-x^2)}\right) = 12\sum_{n=0}^{\infty}(-1)^n(-x^2)^n$$

$$= \sum_{n=0}^{\infty} 12x^{2n}$$

for testing convergence

$$\lim_{n\to\infty}\left|\frac{12x^{2(n+1)}}{12x^{2n}}\right| = \lim_{n\to\infty} x^2 = x^2 < 1$$

diverges at both $x=-1$ and $x=1$ (ie $\sum 12$ diverges)

$(-1,1)$ is the interval of convergence

21. Use the power series

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x)=\frac{10}{(x+10)^3}=\frac{d^2}{dx^2}\left(\frac{5}{x+10}\right)$$

$$\frac{5}{x+10}=\frac{1/2}{1+\frac{x}{10}}=\sum_{n=0}(-1)^n\left(\frac{x}{10}\right)^n$$

$\rightarrow$ converges for $\left|\frac{x}{10}\right|<1 \Rightarrow |x|<10$

$$\frac{d}{dx}\frac{5}{x+10}=\sum_{n=1}^{\infty}(-1)^n\frac{n}{10}\left(\frac{x}{10}\right)^{n-1}$$

$$\rightarrow f(x)=\sum_{n=2}^{\infty}(-1)^n\frac{n(n-1)}{100}\left(\frac{x}{10}\right)^{n-2}$$

diverges for both -10 & 10 $\Rightarrow$ $(-10,10)$ is the interval of convergence.

22. Use the power series

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

$$f(x)=\frac{1}{4x^2+1}=\sum(-1)^n(4x^2)^n$$

convergence test $\lim_{n\to\infty}\frac{(4x^2)^{n+1}}{(4x^2)^n}=4x^2<1$

$\Rightarrow -\frac{1}{2}<x<\frac{1}{2}$ for convergence, series diverges for both $x=\pm\frac{1}{2}$

so $\left(-\frac{1}{2},\frac{1}{2}\right)$ is the interval of convergence.

23. Find the Taylor series (centered at c) for the function.

$f(x)=e^{10x},\ c=0$

$$f(x)=\sum_{n=0}^{\infty}\frac{10^n}{n!}x^n$$

$$f^{(n)}(x)=10^n e^{10x}$$

$$f^{(n)}(0)=10^n$$

24. Find the Taylor series (centered at c) for the function.

$f(x)=\sin(x), c=\frac{\pi}{4}$

$f'(x)=\cos(x)$

$f''(x)=-\sin(x)$

$f'''(x)=-\cos(x)$

$f^{(4)}(x)=\sin(x)$ etc

$f(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

$f'(\frac{\pi}{4})=\frac{1}{\sqrt{2}}$

$f''(\frac{\pi}{4})=-\frac{1}{\sqrt{2}}$

$f'''(\frac{\pi}{4})=-\frac{1}{\sqrt{2}}$

etc

$\sin x=\frac{1}{\sqrt{2}}\left(1+\left(x-\frac{\pi}{4}\right)-\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}-\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}\right)$ etc

25. Find the Taylor series (centered at c) for the function.

$f(x)=\ln\left(x^6\right), c=1$

$=6\ln x$

$f'(x)=6x^{-1}$

$f''(x)=-6x^{-2}$

$f'''(x)=12x^{-3}$

etc

$f(1)=0$

$f'(1)=6$

$f''(1)=-6$

$f'''(1)=12$

$f^{(n)}(1)=(-1)^{n+1}6(n-1)!$

$\sum(-1)^{n+1}\frac{6(x-1)^n}{n}$

26. Use the binomial series to find the Maclaurin series for the function.

$f(x)=\frac{1}{\sqrt[10]{1-x}}$

$=(1+(-x))^{-\frac{1}{10}}=1+\left(-\frac{1}{10}\right)(-x)+\frac{\left(-\frac{1}{10}\right)\left(-\frac{11}{10}\right)(-x)^2}{2!}+\frac{\left(-\frac{1}{10}\right)\left(-\frac{11}{10}\right)\left(-\frac{21}{10}\right)(-x)^3}{3!}+\ldots$

$=1+\sum_{n=1}^{\infty}\frac{1\cdot 11\cdot 21\cdots(10n-9)}{10^n n!}x^n$

27. Use the binomial series to find the Maclaurin series for the function.

$f(x)=\sqrt{1+x^4}$

$=(1+x^4)^{\frac{1}{2}}=1+\frac{1}{2}x^4+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x^4)^2}{2!}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x^4)^3}{3!}+\ldots$

$1+\frac{x^4}{2}+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^{4n}$